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PG Sem I

Paper – CC-2, Mathematical Physics

Unit – 2

Topic Laurent series

Laurent's series

Theorem :- If $f(z)$ is analytic and single valued on two concentric circles C_1 and C_2 with centre at z_0 and in the annulus between them, then $f(z)$ can be represented by the Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0) + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \quad \text{--- (1)}$$

where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z')\,dz'}{(z'-z_0)^{n+1}}$

$$b_n = \frac{1}{2\pi i} \int_C (z'-z_0)^{n-1} f(z')\,dz' \quad \text{--- (2)}$$

Instead of (1) Laurent's series may be expressed as

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-z_0)^n \quad \text{--- (3)}$$

where $A_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z')\,dz'}{(z'-z_0)^{n+1}}$

--- (4)

Proof :- Let $f(z)$ be analytic in the annular region between and on two concentric

Circles C_1 and C_2 of radii r_1 and r_2 respectively with centre at z_0.

~~Proof:~~

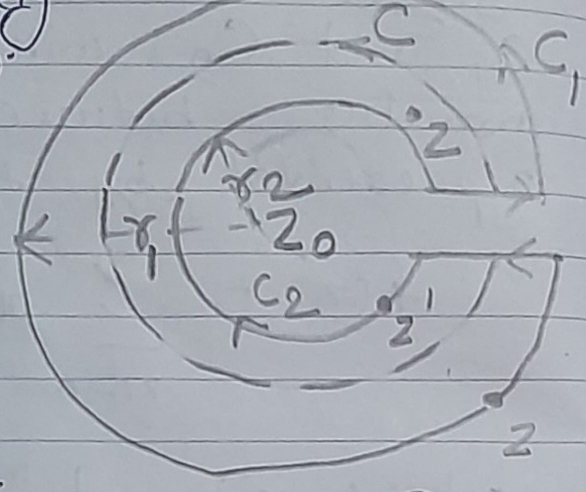

fig ①

Let us draw an imaginary cut line to convert region into a simply connected one; so that Cauchy's integral formula is applicable. From which at each point z in the given annulus it follows that

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(z')\,dz'}{z'-z} \quad \text{— (5)}$$

Here negative sign is introduced because contours C_1 and C_2 are to be traversed in counter clockwise sense. As z lies inside C_1 the first integral in eqn (5) may be developed like that of Taylor's series. Therefore we write

$$\frac{1}{z'-z} = \frac{1}{(z'-z_0)-(z-z_0)} = \frac{1}{(z'-z_0)\left[1-\left(\frac{z-z_0}{z'-z_0}\right)\right]}$$

$$= \frac{1}{(z'-z_0)}\left[1 + \left(\frac{z-z_0}{z'-z_0}\right)_1 + \left(\frac{z'-z_0}{z-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{z'-z_0}\right)^n + \frac{(z-z_0)^{n+1}}{(z'-z_0)^n (z'-z)}\right]$$

Therefore

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{z'-z} = \frac{1}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)} + \frac{(z-z_0)}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^2} + \cdots + \frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}} + R_n(z)$$

where the remainder $R_n(z)$ is given by

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{(z'-z_0)^{n+1}(z'-z)}$$

Now estimating the remainder $R_n(z)$ as in Taylor's series we get

$$R_n(z) \to 0 \text{ as } n \to \infty.$$

Then we obtain

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z')\,dz'}{z'-z} = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{--- (6)}$$

where the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z')\,dz'}{(z'-z_0)^{n+1}} \quad \text{--- (7)}$$

In the second integral of eqn (5); the situation is different, since z' lies outside C_2; therefore, in this case

$\left|\frac{z'-z_0}{z-z_0}\right| < 1$ i.e., we shall now develop

$\frac{1}{z'-z}$ in powers of $\frac{z'-z_0}{z-z_0}$ for the resulting

series to be convergent. Therefore in the second integral we write

$$\frac{1}{z'-z} = \frac{-1}{z-z'} = -\frac{1}{(z-z_0)-(z'-z_0)}$$

$$= -\frac{1}{(z-z_0)\left[1-\left(\frac{z'-z_0}{z-z_0}\right)\right]}$$

$$= \frac{-1}{z-z_0}\left[1-\left(\frac{z'-z_0}{z-z_0}\right)+\left(\frac{z'-z_0}{z-z_0}\right)^2+\cdots+\right.$$

$$\left(\frac{z'-z_0}{z-z_0}\right)^n + \frac{\left[(z'-z_0)/(z-z_0)\right]^{n+1}}{1-\left[(z'-z_0)/(z-z_0)\right]}\Bigg]$$

$$= -\frac{1}{z-z_0}\left[1+\frac{z'-z_0}{z-z_0}+\left(\frac{z'-z_0}{z-z_0}\right)^2+\cdots+\left(\frac{z'-z_0}{z-z_0}\right)^n\right]-\frac{1}{(z-z')}\left(\frac{z'-z_0}{z-z_0}\right)^{n+1}$$

Hence we readily obtain

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(z')\,dz'}{z'-z} = \frac{1}{2\pi i}\left[\frac{1}{(z-z_0)}\int_{C_2} f(z')\,dz' + \frac{1}{(z-z_0)^2}\int_{C_2}(z'-z_0)\,f(z')\,dz' + \cdots + \frac{1}{(z-z_0)^n}\int_{C_2}(z'-z_0)^{n-1} f(z')\,dz' + \frac{1}{(z-z_0)^{n+1}}\int_{C_2}(z'-z_0)^n f(z')\,dz'\right] + Q_n(z) \qquad (8)$$

Where the remainder $Q_n(z)$ is given by

$$Q_n(z) = \frac{1}{2\pi i\,(z-z_0)^{n+1}} \int \frac{(z'-z_0)^{n+1}}{(z-z')} f(z')\,dz' \quad \text{— (9)}$$

To estimate the remainder let $r = |z-z_0|$; then $r_2 < r < r_1$ where $r_2 = |z'-z_0|$ and $|z-z'| \geq r - r_2$.

If M is the maximum value of $f(z')$ on C_2, then it follows from eqn (9) that

$$|Q_n(z)| \leq \frac{1}{2\pi r^{n+1}} \frac{M \cdot 2\pi r_2\, r_2^{n+1}}{r-r_2} = \left(\frac{r_2}{r}\right)^{n+1} \frac{M r_2}{r-r_2}$$

As $\frac{r_2}{r} < 1$, therefore $Q_n(z)$ approaches zero as n tends to infinity. In view of this, eqn (8) can be written in the form

$$-\frac{1}{2\pi i} \int_{C_2} \frac{f(z')\,dz'}{z'-z} = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{— (10)}$$

where the coefficients b_n are given by

$$b_n = \frac{1}{2\pi i} \int_{C_2} (z'-z_0)^{n-1} f(z')\,dz' \quad \text{— (11)}$$

Since z_0 is not a point of the annulus the fun^{n}s $\frac{f(z')}{(z'-z_0)^{n+1}}$ and $(z'-z_0)^{n-1} f(z')$ for all values of n are analytic in the annulus, hence we see that for every contour C within the given annulus with ~~its~~ centre at z_0

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z')\,dz'}{(z'-z_0)^n} \text{ for all } n \geq 0$$

$$b_n = \frac{1}{2\pi i} \int_C (z'-z_0)^{n-1} f(z')\,dz' \text{ for all } n \geq 1$$

Therefore form equ^{n}s 5, 6 and 10, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \quad \text{--- (12)}$$

where a_n and b_n are given by equ^{n}s 7 and 11.

Equ^n 12 represents the Laurent's Series of given analytic fun^n $f(z)$ in its annulus of convergence.

Instead of 12 Laurent's series can be put in a uniform form as follows:

$$f(z) = \sum_{n=-\infty}^{+\infty} A_n (z - z_0)^n \quad \text{——} \quad (13)$$

where the coefficients A_n are given by

$$A_n \frac{1}{2\pi i} \int_C \frac{f(z')\, dz'}{(z' - z_0)^{n+1}} \quad \text{——} \quad (14)$$

where n is any integer positive, negative or zero.

It may be noted that the Laurent's Series of a given funⁿ $f(z)$ in its annulus of convergence is unique. However $f(z)$ may have different Laurent's series in two annuli with the same centre. If a Laurent's series is found by any method, then according to uniqueness property, it must be the Laurent series of given funⁿ in given annulus.